Homework 5
MTH 829 Complex Analysis

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Proposition 0.1 (Exercise VII.10.1)**.** *Let f be a continuous complex valued function on an open subset $G \subset \mathbb{C}$, such that $\int_R f(z)dz = 0$ for every rectangle R with sides parallel to the coordinate axes, with interior in G. Then f is holomorphic.*

Proof. Let $z_0 = x_0 + iy_0 \in G$, and choose $r > 0$ so that $B(z_0, r) \subset G$. For $z \in B(z_0, r)$, define piecewise C^1 curves $\gamma_z^1 = [z_0, x + iy_0] \cup [x + iy_0, x + iy]$ and $\gamma_z^2 = [z_0, x_0 + iy] \cup [x_0 + iy, z]$. Geometrically, γ_z^1 and γ_z^2 are paths from z_0 to z that together trace out a rectangle. Now define two functions

$$g_1(z) = \int_{\gamma_z^1} f(w)dw \qquad g_2(z) = \int_{\gamma_z^2} f(w)dw$$

Let R_z be the rectangle $\gamma_z^1 \cup \gamma_z^2$. By hypothesis, $\int_{R_z} f(w)dw = 0$. To integrate over R_z, we reverse the direction of one of γ_z^1, γ_z^2, so

$$\int_{\gamma_z^1} f(w)dw = \int_{\gamma_z^2} f(w)dw$$

That is, g_1 and g_2 are actually the same function, which we'll call g. We claim that g is holomorphic at z_0 and that $g'(z_0) = f(z_0)$. Viewing g as the integral over γ_z^2 and then differentiating with respect to x gives

$$\begin{aligned}
g(z) &= \int_{\gamma_z^2} f(w)dw = \int_{[z_0, x_0+iy]} f(w)dw + \int_{[x_0+iy, z]} f(w)dw \\
&= \int_{y_0}^{y} f(x_0 + it)i\ dt + \int_{x_0}^{x} f(t + iy)dt \\
\frac{\partial}{\partial x} g(z) &= \frac{\partial}{\partial x} \int_{y_0}^{y} f(x_0 + it)i\ dt + \frac{\partial}{\partial x} \int_{x_0}^{x} f(t + iy)dt \\
&= f(x + iy) + i \frac{\partial}{\partial x} \int_{y_0}^{y} f(x_0 + it)dt \\
&= f(x + iy)
\end{aligned}$$

Viewing g as the integral over γ_z^1 and differentiating with respect to y gives

$$\begin{aligned}
g(z) &= \int_{\gamma_z^1} f(w)dw = \int_{[z_0,x+iy_0]} f(w)dw + \int_{[x+iy_0,z]} f(w)dw \\
&= \int_{x_0}^{x} f(t+iy_0)dt + \int_{y_0}^{y} f(x+it)i\,dt \\
\frac{\partial}{\partial y} g(z) &= \frac{\partial}{\partial y}\int_{x_0}^{x} f(t+iy_0)dt + \frac{\partial}{\partial y}\int_{y_0}^{y} f(x+it)i\,dt \\
&= if(x+iy) + \frac{\partial}{\partial y}\int_{x_0}^{x} f(t+iy_0)dt \\
&= if(x+iy)
\end{aligned}$$

Thus

$$\frac{\partial}{\partial \overline{z}} g = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) g = \frac{1}{2}\big(f(x+iy) + i^2 f(x+iy)\big) = 0$$

Thus g is holomorphic. And we showed along the way that $\frac{\partial}{\partial x} g = f$, so $g' = f$. Hence f is holomorphic. □

Lemma 0.2 (for Exercise VII.10.2)**.** *Let $G \subset \mathbb{C}$ be open and let $f : G \to \mathbb{C}$ be continuous. Let $z_1, z_2 \in G$ so that $[z_1, z_2] \subset \operatorname{Int} G$. Then*

$$\lim_{h\to 0}\int_{[z_1+h,z_2+h]} f(w)dw = \int_{[z_1,z_2]} f(w)dw$$
$$\lim_{h\to 0}\int_{[z_1,z_2+h]} f(w)dw = \int_{[z_1,z_2]} f(w)dw$$

Proof. We begin with the first limit. Parametrize $[z_1, z_2]$ as $\gamma(t) = (1-t)z_1 + tz_2$, and parametrize $[z_1+h, z_2+h]$ as $\gamma_h(t) = \gamma(t) + h$. Then

$$\gamma'(t) = \gamma_h'(t) = z_2 - z_1$$

Thus

$$\int_{\gamma_h} f(w)dw = \int_0^1 f(\gamma(t)+h)(z_2-z_1)dt = (z_2-z_1)\int_0^1 f(\gamma(t)+h)dt$$

Choose a bounded neighborhood of $[z_1, z_2]$ in G. (This exists by compactness of $[z_1, z_2]$.) Then we can choose a smaller, compact neighborhood K of $[z_1, z_2]$. Since f is continuous on K, f is uniformly continuous on this compact neighborhood. Fix $\epsilon > 0$. By uniform continuity of f on K, there exists $\delta > 0$ so that

$$|h| < \delta \implies |f(\gamma(t)) - f(\gamma(t)+h)| < \epsilon$$

for all $t \in [0,1]$. Then for $|h| < \delta$,

$$\begin{aligned}\int_\gamma f(w)dw - \int_{\gamma_h} f(w)dw &= (z_2 - z_1)\left(\int_0^1 f(\gamma(t))dt - \int_0^1 f(\gamma(t)+h)dt\right) \\ &= (z_2 - z_1)\int_0^1 f(\gamma(t)) - f(\gamma(t)+h)dt \\ &\leq (z_2 - z_1)\int_0^1 \epsilon\, dt = (z_2 - z_1)\epsilon\end{aligned}$$

This says precisely that

$$\lim_{|h|\to 0}\int_{\gamma_h} f(w)dw = \int_\gamma f(w)dw$$

Note that $|h| \to 0$ is equivalent to $h \to 0$, so this is exactly what we wanted to prove.

Now we do the second limit, which is pretty much the same argument. Again, parametrize $[z_1, z_2]$ as $\eta(t) = (1-t)z_1 + tz_2$ and now parametrize $[z_1, z_2+h]$ as $\eta(t) = (1-t)z_1 + t(z_2+h) = \eta(t) + th$. Then

$$\int_{\eta_h} f(w)dw = (z_2 - z_1)\int_0^1 f(\eta(t) + th)dt$$

As above, we can choose a compact neighborhood of $[z_1, z_2]$, on which f is uniformly continuous. Fix $\epsilon > 0$. Then there exists $\delta > 0$ such that for all $t \in [0,1]$ we have

$$|h| < \delta \implies |f(\eta(t)) - f(\eta(t)+h)| < \epsilon$$

Since $t \in [0,1]$, $|th| < |h| < \delta$, which implies

$$|f(\eta(t) - f(\eta(t)+th)| < \epsilon$$

Now we repeat the chain of equalities and inequalities of integrals with th instead of h and η instead of γ.

$$\begin{aligned}\int_\eta f(w)dw - \int_{\eta_h} f(w)dw &= (z_2 - z_1)\left(\int_0^1 f(\eta(t))dt - \int_0^1 f(\eta(t)+th)dt\right) \\ &= (z_2 - z_1)\int_0^1 f(\eta(t)) - f(\eta(t)+th)dt \\ &\leq (z_2 - z_1)\int_0^1 \epsilon\, dt = (z_2 - z_1)\epsilon\end{aligned}$$

thus

$$\lim_{|h|\to 0}\int_{\eta_h} f(w)dw = \int_\eta f(w)dw$$

□

Proposition 0.3 (Exercise VII.10.2)**.** *Let f a continuous function in the region $\{z : |z| < 1, \operatorname{Im} z \geq 0\}$, real valued on the segment $(-1,1)$ and holomorphic on the open set $\{z : |z| < 1, \operatorname{Im} z > 0\}$. The f can be extended holomorphically to the open unit disk.*

Proof. Define f on the bottom half of the unit disk by

$$f(z) = \overline{f(\overline{z})}$$

Note that this agrees with the original f on $(-1, 1)$ because f is real-valued there. By Exercise II.8.2, f is holomorphic on $\{z : |z| < 1, \operatorname{Im} z < 0\}$. This extension is continuous on $(-1, 1)$ by the Gluing Lemma for continuous maps. To show that f is now holomorphic on the unit disk, we will show that the integral over every rectangle with sides parallel to the axes is zero. For rectangles contained in either of the half planes $\operatorname{Im} z < 0$ or $\operatorname{Im} z > 0$, this follows from Cauchy's theorem for a convex region, as f is holomorphic off of the real axis. It remains to consider rectangles that intersect the real axis.

First consider a rectangle sitting on the real axis; that is, a rectangle lying in $\operatorname{Im} z \geq 0$. Write it as $R = [z_1, z_2] \cup [z_2, z_3] \cup [z_3, z_4] \cup [z_4, z_1]$ where $z_1, z_2 \in \mathbb{R}$. For $\epsilon > 0$, define R_ϵ to be the rectangle

$$R_\epsilon = [z_1 + i\epsilon, z_2 + i\epsilon] \cup [z_2 + i\epsilon, z_3] \cup [z_3, z_4] \cup [z_4, z_1 + i\epsilon]$$

All of the following integrals are of $f(w)dw$, so we omit the integrand to reduce visual clutter and redudancy.

$$\int_{R_\epsilon} = \int_{[z_1+i\epsilon, z_2+i\epsilon]} + \int_{[z_2+i\epsilon, z_3]} + \int_{[z_3, z_4]} + \int_{[z_4, z_1+i\epsilon]}$$

Applying the previous lemma,

$$\begin{aligned} \lim_{\epsilon \to 0} \int_{[z_1+i\epsilon, z_2+i\epsilon]} &= \int_{[z_1, z_2]} \\ \lim_{\epsilon \to 0} \int_{[z_2+i\epsilon, z_3]} &= \int_{[z_2, z_3]} \\ \lim_{\epsilon \to 0} \int_{[z_4, z_1+i\epsilon]} &= \int_{[z_4, z_1]} \end{aligned}$$

Thus

$$\lim_{\epsilon \to 0} \int_{R_\epsilon} = \int_R$$

Since f is holomorphic on $\operatorname{Im} z > 0$, each integral $\int_{R_\epsilon}$ is zero. Thus $\int_R f(w)dw = 0$. For rectangles sitting under the real axis with one side length along the real axis, the same argument applies. Take limiting rectangles that miss the real axis by ϵ, and apply the previous lemma to get $\lim_{\epsilon \to 0} \int_{R_\epsilon} = \int_R$.

Finally, we need to consider rectangles that cross the real axis. Let R be such a rectangle. Let R_+ and R_- be the rectangles formed by cutting R along the real axis, with R_+ sitting above the real axis and R_- below. By orienting both R_+ and R_- counterclockwise, we see that

$$\int_R = \int_{R_+} + \int_{R_-}$$

which is zero since we just showed that $\int_{R_+} = \int_{R_-} = 0$. We have shown that the integral of our extension of f around any rectangle with sides parallel to the axes and interior in the disk is zero. Thus by Exercise VII.10.1, this extension is holomorphic. □

Proposition 0.4 (Exercise VII.11.1)**.** *Let f be an entire holomorphic function such that for some $k \in \mathbb{Z}$ and $R > 0$, $\left|\frac{f(z)}{z^k}\right|$ is bounded for $|z| > R$. Then f is a polynomial of degree at most k.*

Proof. Since f is entire, the power series $\sum_{n=0}^{\infty} a_n z^n$ converges to $f(z)$ for all $z \in \mathbb{C}$ (where $a_n = f^{(n)}(0)/n!$). Define

$$g(z) = \sum_{n=k+1}^{\infty} a_n z^{n-k}$$

Since the above series has the same radius of convergence as f, the function g is well defined and entire. Then for $z \neq 0$, we have

$$\frac{f(z)}{z^k} = \sum_{n=0}^{\infty} a_n z^{n-k} = \sum_{n=0}^{k} a_n z^{n-k} + g(z) \implies g(z) = \frac{f(z)}{z^k} - \sum_{n=0}^{k} a_n z^{n-k}$$

so applying the triangle inequality gives

$$|g(z)| \leq \left|\frac{f(z)}{z^k}\right| + \left|\sum_{n=0}^{k} a_n z^{n-k}\right| \leq \left|\frac{f(z)}{z^k}\right| + \sum_{n=0}^{k} |a_n||z^{n-k}|$$

Note that for $n < k$ we have

$$|z| > R \implies |z|^{-1} < R^{-1} \implies |z|^{n-k} < R^{n-k}$$

Let M be a bound for $\left|\frac{f(z)}{z^k}\right|$ on $|z| > R$. Then for $|z| > R$,

$$|g(z)| \leq M + \left|\sum_{n=0}^{k} a_n z^{n-k}\right| \leq M + \sum_{n=0}^{k} |a_n||z^{n-k}| \leq M + \sum_{n=0}^{k} |a_n| R^{n-k}$$

Thus g is a bounded entire function, so by Liouville's Theorem g is constant. Since g has constant term zero, this implies that $a_n = 0$ for $n \geq k+1$. Thus the tail of the series for f is zero after k, so

$$f(z) = \sum_{n=0}^{k} a_n z^k$$

thus f is a polynomial of degree at most k. □

Proposition 0.5 (Exercise VII.13.1)**.** *There is no holomorphic function f on the open unit disk so that $f(\frac{1}{n}) = 2^{-n}$ for $n = 2, 3, \ldots$.*

Proof. Suppose there is such an f. By continuity of f,

$$f(0) = \lim_{n\to\infty} f\left(\frac{1}{n}\right) = \lim_{n\to\infty} 2^{-n} = 0$$

thus $f(0) = 0$. This zero must have finite order since f is not the zero function. Thus we can write f locally at zero as $f(z) = z^k g(z)$ where k is the order of the zero at zero, and $g(0) \neq 0$. Note that g is holomorphic. By continuity of g,

$$g(0) = \lim_{n\to\infty} g\left(\frac{1}{n}\right) = \lim_{n\to\infty} n^k f\left(\frac{1}{n}\right) = \lim_{n\to\infty} n^k 2^{-n} = 0$$

Thus $g(0) = 0$, which contradicts $g(0) \neq 0$. Thus no such f exists. □

Proposition 0.6 (Exercise VII.13.2)**.** *Let f be a holomorphic function in the open subset G of $\mathbb{C}$. Let $z_0 \in G$ be a zero of f of order m. Then there is a branch of $f^{\frac{1}{m}}$ in some open disk centered at z_0.*

Proof. As f has a zero of order m at z_0, f can be written as $f(z) = (z - z_0)^m g(z)$ in some open disk centered at z_0, where g is holomorphic and $g(z_0) \neq 0$. Since $g(z_0) \neq 0$, there is a (possibly smaller) open disk centered at z_0 on which $g(z_0) \neq 0$, so there is a branch of $g^{\frac{1}{m}}$ on that disk. Thus on that open disk,

$$\left((z - z_0)g^{\frac{1}{m}}\right)^m = (z - z_0)^m g(z) = f(z)$$

That is to say, $(z - z_0)g^{\frac{1}{m}}$ is a branch of $f^{\frac{1}{m}}$ on some open disk centered at z_0. □

Proposition 0.7 (Exercise VII.14.2a)**.** *Let G be a nonempty connected open subset of $\mathbb{C}$ which is symmetric with respect to the real axis, that is,*

$$G = \{\overline{z} : z \in G\}$$

Let $f : G \to \mathbb{C}$ be holomorphic and suppose that $f(G \cap \mathbb{R}) \subset \mathbb{R}$. Then $f(\overline{z}) = \overline{f(z)}$ for $z \in G$.

Proof. Define $g(z) = \overline{f(\overline{z})}$. By Exercise II.8.2, g is holomorphic on $\{\overline{z} : z \in G\}$, which is G by hypothesis. For $z \in G \cap \mathbb{R}$,

$$g(z) = \overline{f(\overline{z})} = \overline{f(z)} = f(z)$$

since z is real and $f(z)$ is real by hypothesis. Note that since G is symmetric with respect to the real axis and connected, G intersects the real axis in at least one interval. Intervals on the real line clearly contain a limit point of themselves. Hence f and g are holomorphic functions on G that agree on a set with a limit point in G, so $f = g$. Thus

$$f(\overline{z}) = g(\overline{z}) = \overline{f(z)}$$

for all $z \in G$. □

Proposition 0.8 (Exercise VII.14.2b)**.** *Let G be a nonempty connected open subset of $\mathbb{C}$ which is symmetric with respect to the real axis, that is,*

$$G = \{\overline{z} : z \in G\}$$

Let $f : G \to \mathbb{C}$ be holomorphic, and suppose there is a nonempty subinterval $(a, b) \subset \mathbb{R}$ such that $(a, b) \subset G \cap \mathbb{R}$. Then $f(G \cap \mathbb{R}) \subset \mathbb{R}$.

Proof. As in part (a), define $g(z) = \overline{f(\overline{z})}$. Then g is holomorphic on G by Exercise II.8.2. For $z \in (a, b)$,

$$g(z) = \overline{f(\overline{z})} = \overline{f(z)} = f(z)$$

so f and g agree on a set with a limit point in G, so $f = g$. Thus for $z \in G$,

$$f(\overline{z}) = g(\overline{z}) = \overline{f(z)}$$

For $z \in G \cap \mathbb{R}$, we have $z = \overline{z}$, so $f(z) = \overline{f(z)}$. This implies that $f(z)$ is real, since the real line includes all fixed points of the conjugate reflection. □

Proposition 0.9 (Exercise VII.16.1)**.** *Let f be a nonconstant holomorphic function on the connected open subset G of $\mathbb{C}$. Then $|f|$ can attain a local minimum in G only at a zero of f.*

Proof. Suppose $z_0 \in G$ is a local minimum of $|f|$ and suppose that $f(z_0) \neq 0$. Then since f is continuous, there exists $r_1 > 0$ so that $f(z) \neq 0$ on $B(z_0, r_1)$. Since z_0 is a local minimum of $|f|$, there exists r_2 so that $|f(z_0)| \leq |f(z)|$ on $B(z_0, r_2)$. Let $r = \min(r_1, r_2)$, and consider the function $g(z) = \frac{1}{f(z)}$ on $B(z_0, r)$. This is well defined because f, and hence $|f|$ is nonzero on $B(z_0, r)$. Since $|f|$ attains a local minimum at z_0, $|g| = \frac{1}{|f|}$ attains a local maximum at z_0. But this contradicts the maximum modulus principal. Thus we conclude that either z_0 is not a local minimum of $|f|$ or $f(z_0) = 0$. □

Proposition 0.10 (Exercise VII.17.2, Pick's Lemma)**.** *Let D be the open unit disk. Let $f : D \to D$ be holomorphic. For $z, w \in D$ we have*

$$\left| \frac{f(z) - f(w)}{1 - f(z)\overline{f(w)}} \right| \leq \left| \frac{z - w}{1 - z\overline{w}} \right|$$

Proof. Fix $w \in D$. Define the fractional linear transformations

$$T_1(z) = \frac{z - w}{\overline{w}z - 1} = \frac{w - z}{1 - \overline{w}z} \qquad T_2(z) = \frac{z - f(w)}{\overline{f(w)}z - 1} = \frac{f(w) - z}{1 - \overline{f(w)}z}$$

By Exercise III.9.4, both T_1 and T_2 map D onto itself. Note that $T_1(w) = 0$, so $T_2 f T_1^{-1}(0) = T_2(f(w)) = 0$, that is $T_2 f T_1^{-1}$ is a holomorphic self map of the open disk with a fixed point at zero. Thus by applying Schwarz's Lemma to the map $T_2 f T_1^{-1}$ we get

$$|T_2 f(z)| = |T_2 f T_1^{-1}(T_1(z))| \leq |T_1(z)|$$

which we can rewrite as

$$\left| \frac{f(z) - f(w)}{1 - f(z)\overline{f(w)}} \right| = \left| \frac{f(w) - f(z)}{1 - \overline{f(w)}f(z)} \right| \leq \left| \frac{z - w}{\overline{w}z - 1} \right| = \left| \frac{z - w}{1 - z\overline{w}} \right|$$

which is precisely the inequality we wanted. □

Proposition 0.11 (Exercise VII.17.2)**.** *Let D be the open unit disk, and let $f : D \to D$ be a holomorphic map. Then for $w \in D$ we have*

$$|f'(w)| \leq \frac{1 - |f(w)|^2}{1 - |w|^2}$$

Proof. By Exercise VII.17.1,

$$\left| \frac{f(z) - f(w)}{z - w} \right| \leq \left| \frac{1 - f(z)\overline{f(w)}}{1 - z\overline{w}} \right|$$

Now note that

$$|f'(w)| = \left| \lim_{z \to w} \frac{f(z) - f(w)}{z - w} \right| = \lim_{z \to w} \left| \frac{f(z) - f(w)}{z - w} \right|$$

Taking the limit as $z \to w$ on both sides of our inequality,

$$|f'(w)| \leq \lim_{z \to w} \left| \frac{1 - f(z)\overline{f(w)}}{1 - z\overline{w}} \right| = \left| \frac{1 - f(w)\overline{f(w)}}{1 - w\overline{w}} \right| = \left| \frac{1 - |f(w)|^2}{1 - |w|^2} \right| = \frac{1 - |f(w)|^2}{1 - |w|^2}$$

We can drop the absolute value bars at the end because $w, f(w) \in D$, so $|w|, |f(w)| < 1$. $\square$